

Cartier isomorphism for unital associative algebras

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Contents

1 Preliminaries.	4
2 Cyclic homology of algebras.	9
3 Trace maps.	12
4 Cartier isomorphism.	17
5 Comparison.	21

Introduction.

For a smooth affine scheme $X = \operatorname{Spec} A$ of finite type over a perfect field k of positive characteristic, the *Cartier isomorphism* identifies the de Rham cohomology of X with the spaces of differential forms. More precisely, for any $i \geq 0$, let $\Omega^\bullet(X)$ be space of global i -forms on X , and let

$$B^\bullet(X), Z^\bullet(X) \subset \Omega^\bullet(X)$$

be the subspaces of exact resp. closed forms. Then there exists a canonical isomorphism

$$C : Z^\bullet(X)/B^\bullet(X) \cong \Omega^\bullet(X).$$

This isomorphism is Frobenius-semilinear with respect to the Frobenius endomorphism of the field k . It is functorial and compatible with localizations, so that it induces an isomorphism of Zariski sheaves on X (and in this from, it generalizes to non-affine schemes).

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The classic theorem of Hochschild, Kostant and Rosenberg identifies the spaces $\Omega^\bullet(X)$ with the Hochschild homology groups $HH_\bullet(A)$ of the algebra A – for any $i \geq 0$, we have a canonical isomorphism

$$\Omega^i(X) \cong HH_i(A).$$

The de Rham differential d coincides under this identification with the Connes-Tsygan differential $B : HH_\bullet(A) \rightarrow HH_{\bullet+1}(A)$. However, both the Hochschild homology groups and the Connes-Tsygan differential exist in a larger generality – they are defined for an arbitrary unital associative algebra A . It is natural to ask whether the Cartier isomorphism exists in this larger generality.

The goal of this paper is to give a positive answer to this question, provided the characteristic of the base field is not 2. Namely, for any associative unital k -algebra A , we construct functorial groups $BHH_\bullet(A)$, $ZHH_\bullet(A)$ and maps

$$BHH_\bullet(A) \xrightarrow{\xi} ZHH_\bullet(A) \xrightarrow{\zeta} HH_\bullet(A) \xrightarrow{\beta} BHH_{\bullet+1}(A)$$

such that the Connes-Tsygan differential factors as $B = \zeta \circ \xi \circ \beta$, and the map ξ fits into a natural long exact sequence

$$BHH_\bullet(A) \xrightarrow{\xi} ZHH_\bullet(A) \xrightarrow{C} HH_\bullet(A) \longrightarrow ,$$

with a certain natural Frobenius-semilinear map C . This map C is our non-commutative generalization of the Cartier isomorphism. We then prove that in the Hochschild-Kostant-Rosenberg case – that is, A is commutative of finite type and $X = \text{Spec } A$ is smooth – our generalized objects reduce to the classic ones: $BHH_\bullet(A)$ and $ZHH_\bullet(A)$ become $B^\bullet(X)$ and $Z^\bullet(X)$, the maps α and β are the natural embeddings, and C is the usual Cartier isomorphism.

Note that while the Cartier isomorphisms of the individual cohomology groups are the most one can get in the general case, stronger results can be obtained if one imposes restrictions on the variety X . For example, if one assumes that X is liftable to the ring $W_2(k)$ of second Witt vectors of k , and that $\dim X$ is less than $\text{char } k$, then C can be lifted to an isomorphism

$$F_*\Omega^\bullet(X) \cong \bigoplus_i \Omega^i(X^{(1)})[-i]$$

in the derived category of coherent sheaves on the Frobenius twist $X^{(1)}$, where $F : X \rightarrow X^{(1)}$ is the Frobenius map. This forms the basis of the

famous paper [DI] of P. Deligne and L. Illusie. A non-commutative generalization of the Deligne-Illusie isomorphism has been already constructed in [K] (under the assumptions that A lifts to $W_2(k)$, and the homological dimension of the category of A -bimodules is less than $2 \operatorname{char} k$). Therefore in this paper, we concentrate on the results that require no assumptions on the associative algebra A .

Another topic we avoid in this paper is the so-called *conjugate spectral sequence* – namely, the spectral sequence generated by the canonical filtration on the de Rham complex $\Omega^\bullet(X)$. In the classic case, this spectral sequence converges to de Rham cohomology of X , and the Cartier isomorphism identifies the first term of the spectral sequence with the Frobenius twist of the Hodge cohomology of X . The corresponding statement for cyclic homology is also true, in a sense, but it is significantly more delicate because of possible convergence issues – in effect, what the conjugate spectral sequence converges to is not the usual periodic cyclic homology but a different homology theory. We note that already in the commutative case, a similar phenomenon occurs when one studies singular schemes and derived de Rham complexes, see e.g. the recent work of A. Beilinson [Be] and B. Bhatt [Bh]. We will return to this elsewhere.

The paper is organized as follows. In Section 1, we recall the necessary generalities on cyclic categories, cyclic homology and so on. The material is pretty standard. We follow the exposition given previously in [K] and use more-or-less the same notation and conventions. In Section 2, we recall how cyclic homology works for associative algebras – again, this material is very standard. Section 3 is concerned with a slightly non-standard part of the cyclic homology story, so we give complete proofs. Having finished with the preliminaries, we can turn to our subject. In Section 4, we define our non-commutative Cartier map C and the whole package that comes with it – the groups $BHH_\bullet(A)$, $ZHH_\bullet(A)$ and all the natural maps between them. The map C appears at the very end of the Section in Definition 4.6. Finally, in Section 5, we prove the comparison result stating that in the HKR case, our generalized Cartier maps reduces to the classical one (this is Proposition 5.1).

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1 Preliminaries.

Our approach to cyclic homology is based on homology of small categories, as in [C] and [FT]. Our notation is as follows. For any small category I and ring R , we denote by $\text{Fun}(I, R)$ the category of functors from I to the category of left R -modules. This is an abelian category. We denote its derived category by $\mathcal{D}(I, R)$. If $I = \text{pt}$ is the point category, then $\text{Fun}(\text{pt}, R)$ is the category of left R -modules, and $\mathcal{D}(\text{pt}, R)$ is its derived category $\mathcal{D}(R)$. For any functor $\gamma : I' \rightarrow I$ between small categories, we denote by $\gamma^* : \text{Fun}(I, R) \rightarrow \text{Fun}(I', R)$ the natural pullback functor, and we denote by $\gamma_!, \gamma_* : \text{Fun}(I', R) \rightarrow \text{Fun}(I, R)$ its left and right-adjoint, namely, the left and right Kan extension along γ . The derived functors $L^\bullet \gamma_!, R^\bullet \gamma_* : \mathcal{D}(I', R) \rightarrow \mathcal{D}(I, R)$ are left resp. right-adjoint to the pullback functor $\gamma^* : \mathcal{D}(I, R) \rightarrow \mathcal{D}(I', R)$. The *homology complex* of a small category I with coefficients in an object $E \in \mathcal{D}(I, R)$ is then given by

$$C_\bullet(I, E) = L^\bullet \tau_! E \in \mathcal{D}(R),$$

where $\tau : I \rightarrow \text{pt}$ is the projection to the point. The homology groups $H_\bullet(I, E)$ are the homology groups of the complex $C_\bullet(I, R)$.

The small category one needs for applications to cyclic homology is the *cyclic category* Λ introduced by A. Connes [C]. It has many equivalent definitions, for example the following:

- objects of Λ are finite cellular decompositions of the circle S^1 , with $[n] \in \Lambda$ standing for the decomposition with n cells of dimension 0,
- morphisms from $[n]$ to $[m]$ are homotopy classes of continuous cellular maps $f : S^1 \rightarrow S^1$ of degree 1 such that the universal covering map $\tilde{f} : \mathbb{R} \rightarrow \mathbb{R}$ is order-preserving.

For any $[n] \in \Lambda$, 0-cells in the corresponding cellular decomposition are called *vertices*, and the set of vertices is denoted $V([n])$. For alternative definitions of Λ , see e.g. [L].

We will also need the following modification of the category Λ . Fix an integer $n \geq 1$, and consider the n -fold covering map $S^1 \rightarrow S^1$. Then a

cellular decomposition of S^1 corresponding to an object $[m] \in \Lambda$ induces a cellular decomposition of its n -fold cover, thus gives an object $[nm] \in \Lambda$, and the deck transformation gives a map $\sigma : [nm] \rightarrow [nm]$ in Λ such that $\sigma^n = \text{id}$. The *category* Λ_n is the category of such objects $[nm] \in \Lambda$ and morphisms between them that commute with σ . Forgetting σ then gives a functor

$$i_n : \Lambda_n \rightarrow \Lambda,$$

and on the other hand, taking the quotient by σ gives a functor

$$(1.1) \quad \pi^n : \Lambda_n \rightarrow \Lambda.$$

Objects of Λ_n are numbered by integers $m \geq 1$; we denote the object corresponding to m by $[m] \in \Lambda$, so that $\pi^n([m]) = [m]$ and $i_n([m]) = [mn]$. We note that the functor π^n of (1.1) is a bifibration in the sense of [G]. Each fiber of this bifibration is the groupoid \mathbf{pt}_n with one object with automorphism group $\mathbb{Z}/n\mathbb{Z}$ generated by σ . Of course, for $n = 1$, we have $\Lambda_1 = \Lambda$ and $i_1 = \pi^1 = \text{id}$.

As usual, we denote by Δ the category of finite non-empty totally ordered sets, with $[n] \in \Delta$ being the set with n elements, and we denote by Δ^o the opposite category. One shows – see e.g. [K, Subsection 1.5] that uses exactly the same notation as this paper – that there exists a natural functor

$$j : \Delta^o \rightarrow \Lambda$$

sending $[n] \in \Delta^o$ to $[n] \in \Lambda$ (in effect, Δ^o is equivalent to the category $[1] \setminus \Lambda$ of objects $[n] \in \Lambda$ equipped with a map $[1] \rightarrow [n]$, and j is the functor that forgets the map). For any $n \geq 1$, we also have an analogous functor $j_n : \Delta^o \rightarrow \Lambda_n$, and $\pi^n \circ j_n \cong j$.

Definition 1.1. For any $n \geq 1$, any ring R , and any object $E \in \mathcal{D}(\Lambda_n, R)$, the *Hochschild homology* $HH_\bullet(E)$ resp. the *cyclic homology* $HC_\bullet(E)$ are given by

$$HH_\bullet = H_\bullet(\Delta^o, j_n^* E), \quad HC_\bullet(E) = H_\bullet(\Lambda_n, E).$$

For any $E \in \text{Fun}(\Delta^o, R)$, the homology $H_\bullet(\Delta^o, E)$ can be computed by the standard complex $CH_\bullet(E)$ of the simplicial R -module E , namely, $CH_i(E) = E([i+1])$, $i \geq 0$, with the differential given by the alternating sum of the face maps. By abuse of notation, for any $E \in \text{Fun}(\Lambda_n, R)$, we will denote $CH_\bullet(E) = CH_\bullet(j_n^* E)$, and call it the *Hochschild homology complex* of the object E .

To analyze cyclic homology, it is convenient to do the following. For any $[n] \in \Lambda$, denote by $K_\bullet([n])$ the cellular chain complex of the circle S^1 computing its homology with coefficients in \mathbb{Z} . This is functorial with respect to cellular maps, and the homology does not depend on the choice of a cellular decomposition, so that we obtain an exact sequence

$$(1.2) \quad \mathbb{Z} \xrightarrow{\kappa_1} \mathbb{K}_1 \longrightarrow \mathbb{K}_0 \xrightarrow{\kappa_0} \mathbb{Z}$$

in $\text{Fun}(\Lambda, \mathbb{Z})$, where \mathbb{Z} stands for the constant functor. For any $n \geq 1$, we can apply the pullback functor i_n^* and obtain an analogous exact sequence in $\text{Fun}(\Lambda_n, \mathbb{Z})$. For any $E \in \text{Fun}(\Lambda_n, R)$, denote

$$\mathbb{K}_i(E) = i_n^* \mathbb{K}_i \otimes E, \quad i = 0, 1.$$

Then $\mathbb{K}_\bullet(E)$ is a complex in $\text{Fun}(\Lambda_n, R)$ of length 2, with homology objects in degrees 0 and 1 canonically identified with E .

Lemma 1.2. (i) *For any $n \geq 1$ and $E \in D(\Lambda, R)$, the adjunction maps*

$$h_n : HH_\bullet(i_n^* E) \rightarrow HH_\bullet(E), \quad c_n : HC_\bullet(i_n^* E) \rightarrow HC_\bullet(E)$$

are isomorphisms.

(ii) *For any $n \geq 1$ and $E \in \text{Fun}(\Lambda_n, R)$, we have $HH_\bullet(\mathbb{K}_1(E)) = 0$, and we have a natural isomorphism $HH_\bullet(E) \cong HC_\bullet(\mathbb{K}_0(E))$, so that we have an identification*

$$(1.3) \quad HH_\bullet(E) \cong HC_\bullet(\mathbb{K}_\bullet(E)).$$

Proof. The isomorphism h_n in (i) is the so-called *edgewise subdivision* isomorphism that goes back to Quillen and Segal; for an independent proof of (i) with exactly the same notation as here, see [K, Lemma 1.14]. For (ii), see [K, Subsection 1.6] and specifically [K, Lemma 1.10]. \square

Now for any $E \in \text{Fun}(\Lambda_n, R)$, consider the natural maps

$$(1.4) \quad \kappa_0 : \mathbb{K}_0(E) \longrightarrow E, \quad \kappa_1 : E \longrightarrow \mathbb{K}_1(E)$$

induced by the maps (1.2), and let

$$(1.5) \quad B = \kappa_1 \circ \kappa_0 : \mathbb{K}_0(E) \rightarrow E \rightarrow \mathbb{K}_1(E)$$

be their composition. We can form a natural bicomplex

$$(1.6) \quad \begin{array}{ccccc} & & & & \mathbb{K}_0(E) \\ & & & & \uparrow \\ & & \mathbb{K}_0(E) & \xrightarrow{B} & \mathbb{K}_1(E) \\ & & \uparrow & & \\ & \mathbb{K}_0(E) & \xrightarrow{B} & \mathbb{K}_1(E) & \\ & \uparrow & & & \\ \xrightarrow{B} & \mathbb{K}_1(E) & & & \end{array}$$

The total complex of this bicomplex is a resolution of the object E . If we denote by u the endomorphism of (1.6) obtain by shifting to the left and downward by 1 term, then by virtue of (1.3), one of the two standard spectral sequences associated to a bicomplex reads as

$$(1.7) \quad HH_*(E)[u] \Rightarrow HC_*(\Lambda_l, E),$$

where the left-hand side is shorthand for “formal polynomials in one variable u of homological degree 2 with coefficients in $HH_*(E)$ ”. This is known as the *Hochschild-to-cyclic*, or *Hodge-to-de Rham* spectral sequence. The first non-trivial differential

$$(1.8) \quad B : HH_*(E) \rightarrow HH_{*+1}(E)$$

is known as the *Connes-Tsygan differential*, or sometimes *Rinehart differential*. In terms of the identification (1.3), this differential is induced by the map

$$(1.9) \quad B : \mathbb{K}_*(E) \rightarrow \mathbb{K}_*(E)[-1]$$

which is in turn induced by the natural map B of (1.5) (this is why we use the same notation for all three). Moreover, B can be lifted to a map of complexes

$$(1.10) \quad B : CH_*(E) \rightarrow CH_*(E)[-1]$$

so that $B^2 = 0$ (see e.g. [L]).

If the ring R is commutative, then the categories $\text{Fun}(\Lambda, R)$, $\text{Fun}(\Delta^o, R)$ acquire natural pointwise tensor products, and their derived categories in

turn acquires the derived tensor product $\overset{\mathbb{L}}{\otimes}$. It is well-known that the Hochschild homology functor is multiplicative: for any two objects $E, E' \in \mathcal{D}(\Lambda, R)$, we have a natural Künneth quasiisomorphism

$$(1.11) \quad CH_{\bullet}(E) \overset{\mathbb{L}}{\otimes} CH_{\bullet}(E') \cong CH_{\bullet}(E \overset{\mathbb{L}}{\otimes} E'),$$

where by abuse of notation we use $CH_{\bullet}(-)$ to denote the corresponding object in the derived category $\mathcal{D}(R)$.

Lemma 1.3. *The Connes-Tsygan differential B of (1.8) is compatible with the Künneth quasiisomorphism (1.11) – that is, we have*

$$B_{E \overset{\mathbb{L}}{\otimes} E'} = B_E \otimes \text{id} + \text{id} \otimes B_{E'}.$$

This is very well-known if R contains \mathbb{Q} . However, we will need the general case. Since we could not find a convenient reference, we give a proof.

Definition 1.4. A *mixed complex* in an abelian category \mathcal{C} is a pair $\langle \mathbb{K}_{\bullet}, B \rangle$ of a complex \mathbb{K}_{\bullet} in \mathcal{C} and a map $B : \mathbb{K}_{\bullet} \rightarrow \mathbb{K}_{\bullet}[-1]$ such that $B^2 = 0$. A map of mixed complexes is a *quasiisomorphism* if it is a quasiisomorphism of the underlying complexes in \mathcal{C} .

Example 1.5. For any ring R and any $E \in \text{Fun}(\Lambda, R)$, the complex $\mathbb{K}_{\bullet}(E)$ with the differential B of (1.9) is a mixed complex in $\text{Fun}(\Lambda, R)$.

Inverting quasiisomorphisms of mixed complexes, we obtain a triangulated category $\mathcal{D}_{\text{mix}}(\mathcal{C})$. If \mathcal{C} is a tensor category, then one defines the tensor product of two mixed complexes by

$$\langle \mathbb{K}_{\bullet}, B \rangle \otimes \langle K'_{\bullet}, B' \rangle = \langle \mathbb{K}_{\bullet} \otimes K'_{\bullet}, B \otimes \text{id} + \text{id} \otimes B' \rangle,$$

and the derived functor of this tensor product obviously turns $\mathcal{D}_{\text{mix}}(\mathcal{C})$ into a tensor triangulated category (at least when every complex in \mathcal{C} has a homotopically flat replacement, and this is the only case that we will need). If \mathcal{C}' is a module category over \mathcal{C} , then the same formula defines an external tensor product $\langle \mathbb{K}_{\bullet}, B \rangle \otimes \langle K'_{\bullet}, B' \rangle$ of a mixed complex $\langle \mathbb{K}_{\bullet}, B \rangle$ in \mathcal{C} and a mixed complex $\langle K'_{\bullet}, B' \rangle$ in \mathcal{C}' .

Example 1.6. For any ring R and any object $E \in \text{Fun}(\Lambda, R)$, the complex $CH_{\bullet}(E)$ with the differential B of (1.10) is an object in $\mathcal{D}_{\text{mix}}(R)$.

Definition 1.7. Assume that an abelian category \mathcal{C} has countable products. The *truncation* $\mathrm{tr}(\langle \mathbb{K}_\bullet, B \rangle)$ of a mixed complex $\langle \mathbb{K}_\bullet, B \rangle$ in \mathcal{C} is the product-total complex of the bicomplex

$$\mathbb{K}_\bullet \xrightarrow{B} \mathbb{K}_\bullet[-1] \xrightarrow{B} \mathbb{K}_\bullet[-2] \xrightarrow{B} \dots$$

in \mathcal{C} .

Proof of Lemma 1.3. For any mixed complex $\langle \mathbb{K}_\bullet, B \rangle$ of R -modules, let $\lambda(\langle \mathbb{K}_\bullet, B \rangle)$ be the complex in $\mathrm{Fun}(\Lambda, R)$ given by

$$\lambda(\langle \mathbb{K}_\bullet, B \rangle) = \mathrm{tr}(\langle \mathbb{K}_\bullet, B \rangle \otimes \mathbb{K}_\bullet(R)),$$

where R in the right-hand side is shorthand for the constant functor with value R . Then since $\mathbb{K}_\bullet(R)$ is a complex of flat objects in $\mathrm{Fun}(\Lambda, R)$, λ respects quasiisomorphisms and descends to a functor

$$(1.12) \quad \lambda : \mathcal{D}_{\mathrm{mix}}(R) \rightarrow \mathcal{D}(\Lambda, R).$$

Moreover, one checks easily that there exists a quasiisomorphism

$$\mathbb{K}_\bullet(R) \cong \mathrm{tr}(\mathbb{K}_\bullet(R) \otimes \mathbb{K}_\bullet(R)),$$

and this turns λ into a triangulated tensor functor. The Hochschild homology functor $CH_\bullet(-)$ of Example 1.6 is left-adjoint to λ . Since λ is tensor, $CH_\bullet(-)$ is pseudotensor by adjunction, so that we have a natural functorial map

$$CH_\bullet(E \overset{\mathrm{L}}{\otimes} E') \rightarrow CH_\bullet(E) \overset{\mathrm{L}}{\otimes} CH_\bullet(E').$$

This map is automatically compatible with the Connes-Tsygan differentials, and it is inverse to the Künneth quasiisomorphism (1.11). \square

Remark 1.8. In fact, λ is a full embedding of tensor triangulated categories, but we will not need this.

2 Cyclic homology of algebras.

Assume given a commutative ring k . To any associative unital algebra A over k one associates a canonical object $A_\sharp \in \mathrm{Fun}(\Lambda, k)$ as follows:

- on objects, $A_\sharp([n]) = A^{\otimes_k V([n])} = A^{\otimes_k n}$, with copies of A numbered by elements $v \in V([n])$,

- for any map $f : [n] \rightarrow [m]$, the map

$$A_{\#}(f) : A^{\otimes_k n} = \bigotimes_{v \in V([m])} A^{\otimes_k f^{-1}(v)} \rightarrow A^{\otimes m}$$

is given by

$$(2.1) \quad A_{\#}(f) = \bigotimes_{v \in V([m])} m_{f^{-1}(v)},$$

where $m_{f^{-1}(v)} : A^{\otimes_k f^{-1}(v)} \rightarrow A$ is the map which multiplies the entries in the natural clockwise order on the vertices $v' \in f^{-1}(v) \subset S^1$.

Assume that A is flat as a k -module. Then by definition, the *cyclic homology* $HC_{\bullet}(A)$ is given by

$$HC_{\bullet}(A) = HC_{\bullet}(A_{\#}).$$

The *Hochschild homology* $HH_{\bullet}(A, M)$ with coefficients in an A -bimodule M is given by

$$HH_{\bullet}(A) = \text{Tor}_{\bullet}^{A^{opp} \otimes_k A}(A, M);$$

explicitly, in degree 0 we have

$$(2.2) \quad HH_0(A, M) \cong M / \{am - ma \mid a \in A, m \in M\}.$$

To simplify notation, one denotes $HH_{\bullet}(A) = HH_{\bullet}(A, A)$. One easily shows using the bar resolution that we have a natural identification

$$HH_{\bullet}(A) \cong HH_{\bullet}(A_{\#}).$$

Under these identifications, (1.7) reads as

$$(2.3) \quad HH_{\bullet}(A)[u] \Rightarrow HC_{\bullet}(A).$$

Note that by Lemma 1.2 (i), one can equally well replace Λ with Λ_n and $A_{\#}$ with $i_n^* A_{\#}$; the resulting spectral sequence is the same.

In practice, the Hochschild homology groups $HH_{\bullet}(A)$ can be computed by the Hochschild homology complex $CH_{\bullet}(A_{\#})$, denoted $CH_{\bullet}(A)$ to simplify notation; its terms are given by

$$CH_i(A) = A^{\otimes_k i+1},$$

with a certain differential usually denoted by b . The Connes-Tsygan differential B can be lifted to the level of complexes, as in (1.10). Explicitly, we have

$$(2.4) \quad \begin{aligned} B(a_0 \otimes \cdots \otimes a_i) &= \sum_{j=0}^i 1 \otimes a_j \otimes a_{j+1} \otimes \cdots \otimes a_i \otimes a_0 \otimes \cdots \otimes a_{j-1} \\ &\quad - \sum_{j=0}^i a_j \otimes a_{j+1} \otimes \cdots \otimes a_i \otimes a_0 \otimes \cdots \otimes a_{j-1} \otimes 1. \end{aligned}$$

Alternatively, one can fix an integer $n \geq 1$ and apply the edgewise subdivision isomorphism h_n of Lemma 1.2 (i). This gives a different complex $CH_{\bullet}^{(n)}(A) = CH_{\bullet}(i_n^* A_{\sharp})$ with terms

$$CH_i^{(n)}(A) = A^{\otimes_k n(i+1)}$$

computing the same Hochschild homology groups $HH_{\bullet}(A)$. The isomorphism h_n can also be lifted to the level of complexes; on $CH_i^{(n)}(A)$, it is given by

$$(2.5) \quad h_n(a_0^1 \otimes \cdots \otimes a_i^1 \otimes \cdots \otimes a_0^n \otimes \cdots \otimes a_i^n) = a_0^1 \cdots a_i^1 \cdots a_0^n \otimes a_1^n \otimes \cdots \otimes a_i^n,$$

or in words, “multiply the first $(i+1)(n-1)+1$ terms, and keep the last i terms intact”.

In general, both $HH_{\bullet}(A)$ and $HC_{\bullet}(A)$ are just k -modules. However, if A is commutative, then $HH_{\bullet}(A)$ becomes commutative graded k -algebras, with the product induced by the Künneth quasiisomorphism and the natural algebra map $A \otimes A \rightarrow A$. By Lemma 1.3, the Connes-Tsygan differential B is a derivation with the respect to this algebra structure.

The main comparison theorem for Hochschild homology in the commutative case is the classic result of Hochschild, Kostant, and Rosenberg.

Theorem 2.1. *Assume that k is a field, the algebra A is commutative and finitely generated over k , and that $X = \text{Spec } A$ is smooth. Then we have natural isomorphisms*

$$(2.6) \quad HH_i(A) \cong H^0(X, \Omega_X^i)$$

for any $i \geq 0$, and these isomorphisms are compatible with multiplication. \square

We want to emphasize that the Hochschild-Kostant-Rosenberg Theorem requires no assumptions on $\text{char } k$. As for cyclic homology, the main result is the following.

Theorem 2.2. *In the assumptions and under the identifications of Theorem 2.1, the Connes-Tsygan differential*

$$B : HH_i(A) \rightarrow HH_{i+1}(A),$$

$i \geq 0$, becomes the de Rham differential $d : \Omega_X^i \rightarrow \Omega_X^{i+1}$.

Proof. This is extremely well-known if $\text{char } k = 0$ (in fact, it was this result which gave rise to cyclic homology as a separate subject). For the convenience of the reader, let us show that the result also holds when $\text{char } k = p > 0$. Indeed, since B is a derivation, it suffices to check it on $HH_0(A) = A$ and $HH_1(A) = \Omega^1(A)$, the module of Kähler differentials. But if $\text{char } k = p$, then for any i , $0 \leq i < p$ the Hochschild-Kostant-Rosenberg isomorphism (2.6) can be lifted to $CH_i(A)$ and expressed by an explicit formula

$$P(a_0 \otimes \cdots \otimes a_i) = \frac{1}{i!} a_0 da_1 \wedge \cdots \wedge a_i.$$

In particular, this is always possible in degrees 0 and 1. Substituting this into (2.4) immediately proves the claim. \square

3 Trace maps.

The main technical tool in our approach to the Cartier map is the projection (1.1); in this section, we analyze its homological properties.

Fix a field k and an integer $p \geq 1$. For a k -vector space M equipped with a representation of the cyclic group $\mathbb{Z}/p\mathbb{Z}$, let $\sigma \in \mathbb{Z}/p\mathbb{Z}$ be the generator, and consider the spaces M^σ , M_σ of invariants and coinvariants. We have two natural functorial maps between them: the map

$$(3.1) \quad e_p : M^\sigma \rightarrow M_\sigma$$

obtained as the composition of the embedding $M^\sigma \subset M$ and the projection $M \rightarrow M_\sigma$, and the trace map

$$(3.2) \quad \text{tr}_p = 1 + \sigma + \cdots + \sigma^{p-1} : M_\sigma \rightarrow M^\sigma$$

obtained by averaging over the group. We have

$$(3.3) \quad \text{tr}_p \circ e_p = p \text{ id} = e_p \circ \text{tr}_p.$$

Moreover, say that a $k[\mathbb{Z}/p\mathbb{Z}]$ -module is *induced* if $M = M' \otimes_k k[\mathbb{Z}/p\mathbb{Z}]$ for some k -vector space M' . Then for an induced M , the map tr_p is an isomorphism.

Now consider the projection $\pi^p : \Lambda_p \rightarrow \Lambda$ of (1.1). As noted in Section 1, π^p is a bifibration with fiber $\mathbf{pt}_p = \mathbf{pt}/(\mathbb{Z}/p\mathbb{Z})$, so that for any $E \in \text{Fun}(\Lambda_p, R)$ and any $[n]$, $E([n])$ is a representation of $\mathbb{Z}/p\mathbb{Z}$. Moreover, by [K, Lemma 1.7] we have

$$(3.4) \quad \pi_!^p(E)([n]) \cong E([n])_\sigma, \quad \pi_*^p(E)([n]) \cong E([n])^\sigma.$$

The maps \mathbf{e}_p and \mathbf{tr}_p are functorial in $[n]$ and give two natural maps

$$(3.5) \quad \mathbf{e}_p : \pi_*^p E \rightarrow \pi_!^p E, \quad \mathbf{tr}_p : \pi_!^p E \rightarrow \pi_*^p E.$$

If the $k[\mathbb{Z}/p\mathbb{Z}]$ -module $E([n])$ is induced for every $[n] \in \Lambda_p$, then the map \mathbf{tr}_p of (3.5) is an isomorphism, and E is acyclic both for the right-exact functor $\pi_!^p$ and for the left-exact functor π_*^p (that is, $L^i \pi_!^p E = \pi_!^p E$, $R^i \pi_*^p E = \pi_*^p E$). In particular, for any $[n] \in \Lambda_p$, $\mathbb{K}_0([n]) \cong \mathbb{K}_1([n]) \cong \mathbb{Z}[\mathbb{Z}/np\mathbb{Z}]$ are free modules over $\mathbb{Z}[\mathbb{Z}/p\mathbb{Z}]$, so that this applies to functors of the form $\mathbb{K}_0(E) = E \otimes \mathbb{K}_0$, $\mathbb{K}_1(E) = E \otimes \mathbb{K}_1$. Therefore we have a natural isomorphism

$$(3.6) \quad \mathbf{tr}_p : \pi_!^p \mathbb{K}_\bullet(E) \cong \pi_*^p \mathbb{K}_\bullet(E)$$

for any $E \in \text{Fun}(\Lambda_p, E)$. Moreover, since $\pi_!^p \mathbb{K}_\bullet(E) \cong L^\bullet \pi_!^p \mathbb{K}_\bullet(E)$, we have natural identifications

$$(3.7) \quad HC_\bullet(\mathbb{K}_\bullet(E)) \cong HC_\bullet(\pi_!^p \mathbb{K}_\bullet(E)) \cong HC_\bullet(\pi_*^p \mathbb{K}_\bullet(E)).$$

By Lemma 1.2 (ii), all these groups are further identified with $HH_\bullet(E)$. The homology of the complex $\pi_!^p \mathbb{K}_\bullet(E)$ in degree 0 coincides with $\pi_!^p E$, with the identification provided by the map $\pi_!^p(\kappa_1) : \pi_!^p \mathbb{K}_0(E) \rightarrow \pi_!^p E$, while by (3.6), the homology of the same complex in degree 1 is $\pi_*^p E$, with the identification provided by the map $\pi_*^p(\kappa_1) : \pi_*^p E \rightarrow \pi_*^p \mathbb{K}_1(E) \cong \pi_!^p \mathbb{K}_1(E)$. Moreover, denote by

$$(3.8) \quad \tilde{B} : \pi_!^p \mathbb{K}_0(E) \rightarrow \pi_!^p \mathbb{K}_1(E)$$

the map obtained as the composition

$$\pi_!^p \mathbb{K}_0(E) \xrightarrow{\pi_!^p(\kappa_0)} \pi_!^p E \xrightarrow{\mathbf{tr}_p} \pi_*^p E \xrightarrow{\pi_*^p(\kappa_1)} \pi_*^p \mathbb{K}_1(E) \xrightarrow{\mathbf{tr}_p^{-1}} \pi_!^p \mathbb{K}_1(E),$$

where \mathbf{tr}_p^{-1} is the inverse to the isomorphism (3.6). Then by the functoriality of the trace map \mathbf{tr}_p , we have $\mathbf{tr}_p \circ \pi_!^p(\kappa_1) = \pi_*^p(\kappa_1) \circ \mathbf{tr}_p$, so that

$$(3.9) \quad \tilde{B} = \mathbf{tr}_p^{-1} \circ \pi_*^p(\kappa_1) \circ \mathbf{tr}_p \circ \pi_!^p(\kappa_0) = \pi_!^p(\kappa_1) \circ \pi_!^p(\kappa_0) = \pi_!^p(B),$$

where $B : \mathbb{K}_1(E) \rightarrow \mathbb{K}_0(E)$ is the natural map (1.5). Under the identifications (3.7), the map \tilde{B} of (3.8) then induces the Connes-Tsygan differential (1.8) on $HH_\bullet(E)$.

Lemma 3.1. *For any $p \geq 1$ and any $E \in \text{Fun}(\Lambda, k)$, we have natural isomorphisms of complexes*

$$\mathbb{K}_\bullet(E) \cong \pi_!^p \mathbb{K}_\bullet(\pi^{p*} E) \cong \pi_*^p \mathbb{K}_\bullet(\pi^{p*} E).$$

Proof. By the projection formula [K, Lemma 1.7], we have $\pi_!^p \mathbb{K}_\bullet(\pi^{p*} E) \cong E \otimes \pi_!^p \pi^{p*} \mathbb{Z}$ and $\pi_*^p \mathbb{K}_\bullet(\pi^{p*} E) \cong E \otimes \pi_*^p \pi^{p*} \mathbb{Z}$, so that it suffices to consider the case $E = \mathbb{Z}$. In this case, the claim immediately follows from the definition of the objects \mathbb{K}_0 and \mathbb{K}_1 . \square

Next, consider the derived functor $L^\bullet \pi_!^p$. We first observe the following. Since π^p is a bifibration, [K, Lemma 1.7] shows that for any $E \in \text{Fun}(\Lambda_p, k)$ and any $n \geq 1$, we also have a natural identification

$$L^\bullet \pi_!^p(E)([n]) \cong H_\bullet(\mathbb{Z}/p\mathbb{Z}, E([n])),$$

a derived version of (3.4), where the group $\mathbb{Z}/p\mathbb{Z}$ acting on $E([n])$ is generated by the automorphism σ . In particular, this applies to the constant functor $k \in \text{Fun}(\Lambda_p, k)$. But the homology of any group with constant coefficients is a coalgebra, and the homology of the group with any coefficients is a comodule over this coalgebra. We claim that for the group $\mathbb{Z}/p\mathbb{Z}$, this can be made to work relatively over the category Λ .

To see this, let $F_\bullet \in \text{Fun}(\Lambda_p, k)$ be any resolution of the constant functor k by functors F_i such that $F_i([n])$ is a free $k[\mathbb{Z}/p\mathbb{Z}]$ -module for any $i \geq 0$, $n \geq 1$ – for example, we can use the resolution (1.6). Then for any $E \in \text{Fun}(\Lambda_p, k)$ and any $i \geq 0$, the product $F_i \otimes E$ has the same property, so that the resolution $F_\bullet \otimes E$ can be used to compute $L^\bullet \pi_{p!}(E)$. Denote

$$\tilde{\Lambda}_p = \Lambda_p \times_\Lambda \Lambda_p,$$

let $\delta : \Lambda_p \rightarrow \tilde{\Lambda}_p$ be the diagonal embedding, and let $\tilde{\pi}^p : \tilde{\Lambda}_p \rightarrow \Lambda$ be the natural projection. Then for any $E \in \text{Fun}(\Lambda_p, k)$, we have

$$\begin{aligned} L^\bullet \tilde{\pi}_!^p(k \boxtimes E) &\cong \tilde{\pi}_!^p(F_\bullet \boxtimes (F_\bullet \otimes E)) \cong \\ &\cong \pi_!^p(F_\bullet) \otimes \pi_!^p(F_\bullet \otimes E) \cong L^\bullet \pi_!^p(k) \otimes L^\bullet \pi_!^p(E). \end{aligned}$$

On the other hand $\delta^*(k \boxtimes E) \cong E$, and the natural projection

$$\pi_!^p \delta^*(F_\bullet \boxtimes (F_\bullet \otimes E)) = \pi_!^p(F_\bullet \otimes F_\bullet \otimes E) \rightarrow \tilde{\pi}_!^p(F_\bullet \boxtimes (F_\bullet \otimes E))$$

induces a map

$$(3.10) \quad \tilde{a} : L^\bullet \pi_!^p(E) \rightarrow L^\bullet \pi_!^p(k) \otimes L^\bullet \pi_!^p(E).$$

This is our coaction map.

Now assume that p is a prime and $\text{char } k = p$. Then we have the following splitting result.

Lemma 3.2. *For any $E \in \text{Fun}(\Lambda, k)$, there exists a natural isomorphism*

$$(3.11) \quad L^\bullet \pi_!^p \pi^{p*} E \cong \mathbb{K}_\bullet(E)[u] = \bigoplus_{n \geq 0} \mathbb{K}_\bullet(E)[2n].$$

Proof. To compute $L^\bullet \pi_!^p(E)$, we can use the resolution (1.6). Then by (3.9), the differential $\pi_!^p(B)$ in the resulting complex is given by the trace map

$$\text{tr}_p = 1 + \sigma + \cdots + \sigma^{p-1} : \pi_!^p \pi^{p*} E \rightarrow \pi_*^p \pi^{p*} E$$

of (3.5). But $\sigma = \text{id}$ on $\pi^{p*} E$, so that the map tr_p is just multiplication by p , thus equal to 0 by assumption. Combining this with the canonical isomorphism of Lemma 3.1, we obtain the desired isomorphism already on the level of complexes. \square

Composing the coaction map \tilde{a} of (3.10) with the projection $L^\bullet \pi_!^p(k) \rightarrow \mathbb{K}_\bullet(k)$ onto the first summand in (3.11), we obtain a canonical functorial map

$$(3.12) \quad a : L^\bullet \pi_!^p(E) \rightarrow L^\bullet \pi_!^p(E) \otimes \mathbb{K}_\bullet(k)$$

in the derived category $\mathcal{D}(\Lambda, k)$.

To proceed further, we need to assume that p is odd, and impose a condition on the object $E \in \text{Fun}(\Lambda_p, k)$. Namely, recall that if $p \neq 2$, the cohomology algebra $H^\bullet(\mathbb{Z}/p\mathbb{Z}, k)$ is given by

$$H^\bullet(\mathbb{Z}/p\mathbb{Z}, k) \cong k[u]\langle \varepsilon \rangle,$$

where in the right-hand side, we have the free graded-commutative algebra on one generator u of degree 2 and one generator ε of degree 1. For any $k[\mathbb{Z}/p\mathbb{Z}]$ -module E , multiplication by u induces an isomorphism

$$H_{i+2}(\mathbb{Z}/p\mathbb{Z}, E) \cong H_i(\mathbb{Z}/p\mathbb{Z}, E)$$

for any $i \geq 1$.

Definition 3.3. A $k[\mathbb{Z}/p\mathbb{Z}]$ -module E is *tight* if the map

$$(3.13) \quad H_{2i+1}(\mathbb{Z}/p\mathbb{Z}, E) \rightarrow H_{2i}(\mathbb{Z}/p\mathbb{Z}, E)$$

given by multiplication by $\varepsilon \in H^1(\mathbb{Z}/p\mathbb{Z}, k)$ is an isomorphism for any $i \geq 1$. An object $E \in \text{Fun}(\Lambda_p, k)$ is *tight* if $E([n])$ is a tight $k[\mathbb{Z}/p\mathbb{Z}]$ -module for any $[n] \in \Lambda_p$.

To see the tightness condition explicitly, note that by (3.3) we have $\mathbf{e}_p \circ \mathrm{tr}_p = 0$ for every $k[\mathbb{Z}/p\mathbb{Z}]$ -module M , so that \mathbf{e}_p induces a map

$$(3.14) \quad \mathrm{Coker} \, \mathrm{tr}_p \rightarrow \mathrm{Ker} \, \mathrm{tr}_p.$$

If one computes the homology groups $H_\bullet(\mathbb{Z}/p\mathbb{Z}, E)$ by the standard periodic complex, then $\mathrm{Ker} \, \mathrm{tr}$ resp. $\mathrm{Coker} \, \mathrm{tr}$ is identified with $H_{2\bullet+1}(\mathbb{Z}/p\mathbb{Z}, E)$ resp. $H_{2\bullet}(\mathbb{Z}/p\mathbb{Z}, E)$, and the map (3.14) is exactly the map (3.13).

Example 3.4. A trivial $k[\mathbb{Z}/p\mathbb{Z}]$ -module k is tight; so is a free module $k[\mathbb{Z}/p\mathbb{Z}]$.

Remark 3.5. Example 3.4 is essentially exhaustive: all indecomposable tight $k[\mathbb{Z}/p\mathbb{Z}]$ -modules are of this form.

We note that since $\varepsilon^2 = 0$, for any tight $k[\mathbb{Z}/p\mathbb{Z}]$ -module E , the map

$$(3.15) \quad H_{2i}(\mathbb{Z}/p\mathbb{Z}, E) \rightarrow H_{2i-1}(\mathbb{Z}/p\mathbb{Z}, E)$$

given by multiplication by ε is equal to 0. We also note that for a tight $E \in \mathrm{Fun}(\Lambda_p, k)$, all the homology objects of the direct image $L^\bullet \pi_{p!}(E)$ in degree ≥ 1 are canonically identified; we will denote this homology object by

$$(3.16) \quad \mathbf{l}(E) \in \mathrm{Fun}(\Lambda, k).$$

We can now prove our main result in this section. Consider the standard t -structure on the derived category $\mathcal{D}(\Lambda, k)$, and let $\tau_{\leq \bullet}, \tau_{[\bullet, \bullet]}$ be the corresponding truncation functors, so that for any integers $m \geq n$ and any $E \in \mathcal{D}(\Lambda, k)$, we have a natural exact triangle

$$\tau_{\leq n}(E) \longrightarrow \tau_{\leq m}(E) \longrightarrow \tau_{[n+1, m]}(E) \longrightarrow$$

in the category $\mathcal{D}(\Lambda, k)$.

Lemma 3.6. *Assume given a tight object $E \in \mathrm{Fun}(\Lambda_p, E)$, and assume that p is odd. Then for any $i \geq 1$, the coaction map a of (3.12) induces a map*

$$\tau_{\leq -2i} L^\bullet \pi_!^p(E) \rightarrow \tau_{\leq -2i} L^\bullet \pi_!^p(E) \otimes \mathbb{K}_\bullet(k),$$

and the composition map

$$\begin{aligned} \tau_{[-2i-1, -2i]} L^\bullet \pi_!^p(E) &\longrightarrow \tau_{[-2i-1, -2i]} L^\bullet \pi_!^p(E) \otimes \mathbb{K}_\bullet(k) \longrightarrow \\ &\longrightarrow L^{2i} \pi_!^p(E) \otimes \mathbb{K}_\bullet(k) \cong \mathbb{K}_\bullet(\mathbf{l}(E)) \end{aligned}$$

is a quasiisomorphism.

Proof. To prove the first claim, it suffices to check that the composition map

$$\tau_{\leq -2i} L^\bullet \pi_!^p(E) \rightarrow L^\bullet \pi_!^p(E) \otimes \mathbb{K}_\bullet(k) \rightarrow \tau_{\geq -2i+1} L^\bullet \pi_!^p(E) \otimes \mathbb{K}_\bullet(k)$$

is equal to 0. The left-hand side is in $\mathcal{D}_{\leq -2i}(\Lambda, k)$ with respect to the standard t -structure, and the right-hand side is in $\mathcal{D}^{\geq -2i}(\Lambda, k)$; therefore it suffices to check that the map is equal to 0 on homology objects of degree $2i$. This can be checked after evaluating at every $[n] \in \Lambda$, and the corresponding map is the map (3.15) which is indeed equal to 0 for a tight E . Analogously, to prove the second claim, it suffices to check that the map is an isomorphism in homological degrees $2i$ and $2i + 1$; this follows immediately from the definition of a tight object. \square

4 Cartier isomorphism.

We can now present the construction of our generalized Cartier isomorphism. We start with some linear algebra. Assume that our field k of positive characteristic $p = \text{char } k > 0$ is perfect, and let M be a k -vector space. Let the group $\mathbb{Z}/p\mathbb{Z}$ act on the p -th tensor power $M^{\otimes p}$ so that the generator $\sigma \in \mathbb{Z}/p\mathbb{Z}$ acts by the cyclic permutation of order p . Consider the trace map

$$(4.1) \quad \text{tr}_p : (M^{\otimes p})_\sigma \rightarrow (M^{\otimes p})^\sigma$$

of (3.2).

Lemma 4.1. (i) *The $k[\mathbb{Z}/p\mathbb{Z}]$ -module $M^{\otimes p}$ is tight in the sense of Definition 3.3.*

(ii) *Sending $m \in M$ to $m^{\otimes p}$ gives a well-defined additive Frobenius-semilinear map*

$$\psi : M \rightarrow (M^{\otimes p})_\sigma$$

which identifies M with the kernel of the map (4.1). The dual map

$$\widehat{\psi} : (M^{\otimes p})^\sigma \rightarrow M$$

identifies M with the cokernel of map tr .

(iii) *For any $n \geq 0$, we have a commutative diagram*

$$\begin{array}{ccc} (M^{\otimes p^n})^\sigma & \xrightarrow{\widehat{\psi}^n} & M \\ \downarrow & & \downarrow \psi^n \\ M^{\otimes p^n} & \longrightarrow & (M^{\otimes p^n})_\sigma, \end{array}$$

where the left vertical map is the natural embedding, and the bottom map is the natural projection.

Proof. To prove (i), choose a basis in M , so that $M = k[S]$ for some set S . Then

$$(4.2) \quad M^{\otimes p} = k[S^p] = k[S] \oplus k[S^p \setminus S],$$

where S is embedded into S^p as the diagonal. The first summand is trivial, and the second summand is free, and both are tight.

(ii) is actually [K, Lemma 2.3], but let us reproduce the proof for the convenience of the reader. Consider the standard periodic complex

$$\xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}} M^{\otimes p} \xrightarrow{\text{id} - \sigma} M^{\otimes p} \xrightarrow{\text{id} + \sigma + \dots + \sigma^{p-1}}$$

computing the Tate homology $\check{H}_\bullet(\mathbb{Z}/p\mathbb{Z}, M^{\otimes p})$. Define a non-additive map $\psi : M \rightarrow M^{\otimes p}$ by $\psi(m) = m^{\otimes p}$. Then this map goes into the kernel of the differential of the complex (both in odd and even degrees), and it becomes additive after we project onto the first summand in (4.2). Moreover, since k is perfect, it becomes a quasiisomorphism (again both in odd and in even degrees). It remains to notice that the second summand in (4.2) has no Tate homology, so that the corresponding complex is acyclic, and ψ is additive module the image of the differential. Therefore the induced map

$$\psi : M \rightarrow \text{Coker}(\text{id} - \sigma) = (M^{\otimes p})_\sigma$$

is additive and identifies M with the kernel of tr , as required. The second claim follows by duality.

Finally, for (iii), again choose a basis S in M , so that $S^{p^n}/(\mathbb{Z}/p^n\mathbb{Z})$ gives a natural basis both in $(M^{\otimes p^n})^\sigma$ and in $(M^{\otimes p^n})_\sigma$, and note that the composition of the natural embedding and the natural projection is a diagonal operator in this basis, with the entry corresponding to a $\mathbb{Z}/p^n\mathbb{Z}$ -orbit $S' \subset S^{p^n}$ equal to its cardinality $|S'|$. Since $\text{char } k = p$, this is 1 for the one-point orbits, and 0 otherwise. \square

Let now A be an associative algebra over k , and consider the corresponding object $A_\# \in \text{Fun}(\Lambda, k)$ of Section 2 and its pullback $i_p^*(A_\#) \in \text{Fun}(\Lambda_p, k)$ with respect to the embedding $i_p : \Lambda_p \rightarrow \Lambda$. Note that the canonical maps ψ and $\hat{\psi}$ of Lemma 4.1 fit together to give canonical maps

$$(4.3) \quad \psi : A_\# \rightarrow \pi_!^p i_p^* A_\#, \quad \hat{\psi} : \pi_*^p i_p^* A_\# \rightarrow A_\#.$$

Corollary 4.2. *The object $i_p^*(A_\#) \in \text{Fun}(\Lambda_p, k)$ is tight in the sense of Definition 3.3, and the map ψ of (4.3) induces a Frobenius-semilinear isomorphism*

$$A_\# \cong \mathbb{I}(i_p^*(A_\#)).$$

Proof. Immediately follows from Lemma 4.1. \square

To simplify notation, for any $E \in \text{Fun}(\Lambda, k)$ and any $n \geq 1$, let us denote by $\mathbb{K}_\bullet^n(E)$ the length-2 complex in $\text{Fun}(\Lambda, k)$ given by

$$\mathbb{K}_\bullet^n(E) = \pi_!^n \mathbb{K}_\bullet(i_n^* E) \cong \pi_*^n \mathbb{K}_\bullet(i_n^* E).$$

Then (3.7) together with the isomorphism h_n of Lemma 1.2 (i) provide a canonical isomorphism

$$(4.4) \quad HH_\bullet(E) \cong HC_\bullet(\mathbb{K}_\bullet^n(E))$$

for any $E \in \text{Fun}(\Lambda, k)$ and any $n \geq 1$. In particular, for $E = A_\#$ and $n = p$, we have a natural identification

$$HC_\bullet(\mathbb{K}_\bullet^p(A_\#)) \cong HH_\bullet(A_\#) \cong HH_\bullet(A).$$

Definition 4.3. Assume given an object $E \in \text{Fun}(\Lambda, k)$. Then the sub-complexes $B\mathbb{K}_\bullet^p(E), Z\mathbb{K}_\bullet^p(E) \subset \mathbb{K}_\bullet^p(E)$ are the image resp. the kernel of the map

$$\tilde{B} : \mathbb{K}_\bullet^p(E) \rightarrow \mathbb{K}_\bullet^p(E)[-1]$$

of (3.8). For any associative k -algebra A , we denote

$$\begin{aligned} BHH_\bullet(A) &= HC_\bullet(B\mathbb{K}_\bullet^p(A_\#)), \\ ZHH_\bullet(A) &= HC_\bullet(Z\mathbb{K}_\bullet^p(A_\#)). \end{aligned}$$

Alternatively, by Lemma 4.1 (ii) and (3.9), we could define $B\mathbb{K}_\bullet^p(E)$ and $Z\mathbb{K}_\bullet^p(E)$ by saying that on the level of homology, we have

$$H_i(B\mathbb{K}_\bullet^p(E)) = \begin{cases} 0, & i = 0, \\ \text{Ker } \hat{\psi} \subset \pi_*^{p,1} i_p^* A_\# = H_1(\mathbb{K}_\bullet^p(E)), & i = 1, \end{cases}$$

and

$$H_i(Z\mathbb{K}_\bullet^p(E)) = \begin{cases} \text{Im } \hat{\psi} \subset \pi_*^{p,1} i_p^* A_\# = H_0(\mathbb{K}_\bullet^p(E)), & i = 0, \\ \pi_*^{p,1} i_p^* A_\# = H_1(\mathbb{K}_\bullet^p(E)), & i = 1, \end{cases}$$

where ψ and $\hat{\psi}$ are the canonical maps (4.3). In any case, we have

$$B\mathbb{K}_\bullet^p(E) \subset Z\mathbb{K}_\bullet^p(E) \subset \mathbb{K}_\bullet^p(E),$$

and these embeddings induce natural maps

$$(4.5) \quad BHH_{\bullet}(A) \xrightarrow{\xi} ZHH_{\bullet}(A) \xrightarrow{\zeta} HH_{\bullet}(A).$$

In the general case, I do not know whether these maps are injective or not. One obvious observation is that the Connes-Tsygan differential (1.8), being induced by the map \tilde{B} of (3.8), factors through a map

$$(4.6) \quad \beta : HH_{\bullet} \rightarrow BHH_{\bullet+1}(A),$$

and this map actually fits into a long exact sequence

$$(4.7) \quad ZHH_{\bullet}(A) \xrightarrow{\zeta} HH_{\bullet}(A) \xrightarrow{\beta} BHH_{\bullet+1}(A) \longrightarrow$$

This has the following corollary. Denote by $zHH_{\bullet}(Z), bHH_{\bullet}(A) \subset HH_{\bullet}(A)$ the kernel resp. the image of the Connes-Tsygan differential, and denote by $z'HH_{\bullet}(Z), b'HH_{\bullet}(A) \subset HH_{\bullet}(A)$ the image of the map ζ resp. $\zeta \circ \xi$. Then we have inclusions

$$(4.8) \quad bHH_{\bullet}(A) \subset b'HH_{\bullet}(A) \subset z'HH_{\bullet}(A) \subset zHH_{\bullet}(A).$$

Lemma 4.4. *Assume that p is odd. Then there exists a natural Frobenius-semilinear identification*

$$HC_{\bullet}(Z\mathbb{K}_{\bullet}^p(A_{\sharp})/B\mathbb{K}_{\bullet}^p(A_{\sharp})) \cong HH_{\bullet}(A).$$

Proof. Let us compute the direct image $L^{\bullet}\pi_1^p i_p^* A_{\sharp}$ by the standard resolution (1.6). Then we obtain a natural quasiisomorphism

$$Z\mathbb{K}_{\bullet}^p(A_{\sharp})/B\mathbb{K}_{\bullet}^p(A_{\sharp})) \cong \tau_{[-2i-1, -2i]} L^{\bullet}\pi_1^p i_l^* A_{\sharp}.$$

Combining this with Lemma 3.6 and Corollary 4.2, we obtain the desired identification. \square

Corollary 4.5. *For any associative algebra A over a perfect field k of characteristic $p \neq 2$, there exists a canonical long exact sequence*

$$(4.9) \quad BHH_{\bullet}(A) \xrightarrow{\xi} ZHH_{\bullet}(A) \xrightarrow{C} HH_{\bullet}(A) \longrightarrow ,$$

where ξ one of the canonical maps (4.5), and the map C is induced by the isomorphism of Lemma 4.4.

Proof. Clear. \square

Definition 4.6. The *non-commutative Cartier map* for the algebra A is the canonical map C of (4.9).

5 Comparison.

We now need to justify the name “Cartier map” used in Definition 4.6. The comparison result is the following.

Proposition 5.1. *Assume that the algebra A is commutative and finitely generated over the perfect field k , $\text{char } k \neq 2$, and that its spectrum $X = \text{Spec } A$ is smooth. Then the canonical maps ξ, ζ of (4.5) are injective, the map β of (4.6) is surjective, the connecting differential in the long exact sequence (4.9) is trivial, the Hochschild-Kostant-Rosenberg isomorphism induces isomorphisms*

$$BHH_i(A) \cong B\Omega^i(X), \quad ZHH_i(A) \cong Z\Omega^i(X)$$

for every $i \geq 0$, and the non-commutative Cartier map of Definition 4.6 coincides with the classical Cartier isomorphism.

For the proof, we first assume given a monoid G , and consider its group algebra $A = k[G]$. Then the diagonal map $G \rightarrow G^p$ induces a $\mathbb{Z}/p\mathbb{Z}$ -invariant algebra map

$$\varphi : A \rightarrow A^{\otimes p},$$

and this map together with its tensor powers defines a map $\varphi : \pi^{p*} A_{\#} \rightarrow i_p^* A_{\#}$ and a corresponding map

$$\varphi : \pi_!^p \mathbb{K}_{\bullet}(\pi^{p*} A_{\#}) \rightarrow \pi_!^p \mathbb{K}_{\bullet}(i_p^* A_{\#}).$$

By Lemma 3.2, at the level of the derived category, this can be rewritten as

$$(5.1) \quad \Phi : \mathbb{K}_{\bullet}(A_{\#}) \rightarrow \mathbb{K}_{\bullet}^p(A_{\#})$$

(in fact, for any $E \in \text{Fun}(\Lambda, k)$, we have $\pi_!^p \mathbb{K}_{\bullet}(\pi^{p*} E) \cong \mathbb{K}_{\bullet}(E)$ on the nose, but we will not need this).

Lemma 5.2. *The map Φ of (5.1) factors through a map*

$$\Phi : \mathbb{K}_{\bullet}(A_{\#}) \rightarrow Z\mathbb{K}_{\bullet}^p(A_{\#}) \subset \mathbb{K}_{\bullet}^p(A_{\#}),$$

and the induced composition map

$$HH_{\bullet}(A) \xrightarrow{\Phi} ZHH_{\bullet}(A) \xrightarrow{C} HH_{\bullet}(A)$$

is equal to the identity map.

Proof. For any vector space $M = k[S]$ with a base S , the diagonal map $S \rightarrow S^p$ induces a map $\varphi : M \rightarrow M^{\otimes p} = k[S^p]$. This map obviously lands in the subspace $(M^{\otimes p})^\sigma \subset M^{\otimes p}$, and its composition with the canonical map $\widehat{\psi}$ of (4.3) is equal to the identity. Applying this pointwise to the object $A_\#$, we see that the composition

$$A_\# \xrightarrow{\varphi} \pi_*^p i_p^* A_\# \xrightarrow{\widehat{\psi}} \mathbb{I}(i_p^* A_\#) \cong A_\#$$

is also equal to the identity. Applying the identification of Lemma 3.6, we obtain the claim. \square

This immediately implies that in the case $A = k[G]$, the connecting differential in the long exact sequence (4.9) is equal to 0. We in fact have

$$(5.2) \quad ZHH_*(A) \cong \xi(BHH_*(A)) \oplus \Phi(HH_*(A)),$$

and both ξ and Φ are injective.

Next, fix an integer $n \geq 1$, and consider the algebra

$$A_n = k[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$$

of Laurent polynomials in n variables. Then $X = \text{Spec } A_n \cong \mathbb{G}_m^n$ is smooth, and A_n satisfies the assumptions of the Hochschild-Kostant-Rosenberg Theorem and Theorem 2.2. The algebra $\Omega^\bullet(X)$ of differential forms on X is the free graded-commutative A_n -algebra generated by logarithmic derivatives dt_i/t_i , $1 \leq i \leq n$. The easiest way to see the classical Cartier map for X is to consider the multiplicative Frobenius-semilinear map

$$(5.3) \quad C^{-1} : \Omega^\bullet(X) \rightarrow \Omega^\bullet(X)$$

given by $C^{-1}(t_i) = t_i^p$, $C^{-1}(dt_i/t_i) = dt_i/t_i$, $1 \leq i \leq n$. Then C^{-1} actually maps $\Omega^\bullet(X)$ into $Z^\bullet(X) \subset \Omega^\bullet(X)$, and its composition with the classical Cartier map $Z^\bullet(X) \rightarrow \Omega^\bullet(X)$ is equal to the identity.

On the other hand, we have $A_n = k[\mathbb{Z}^n]$, the group algebra of the free abelian group on n generators. Therefore Lemma 5.2 also applies to the algebra A_n , and in particular, we have a natural map

$$\Phi : HH_*(A_n) \rightarrow ZHH_*(A_n).$$

Moreover, fix another integer l , $0 \leq l \leq n$, and consider the subalgebra

$$(5.4) \quad A_{l,n} = k[t_1, \dots, t_l, t_{l+1}, t_{l+1}^{-1}, \dots, t_n, t_n^{-1}] \subset A_n$$

of Laurent polynomials having no poles in the first l variables. Then $Y = \text{Spec } A_{m,n}$ is isomorphic to $\mathbb{G}_a^l \times \mathbb{G}_m^{n-l}$, thus also satisfies the assumptions of the comparison theorems, and the map C^{-1} sends $\Omega^\bullet(Y) \subset \Omega^\bullet(X)$ into itself. On the other hand, $A_{l,n} \cong k[\mathbb{N}^l \times \mathbb{Z}^{n-l}]$, where \mathbb{N} is the monoid of non-negative integers, so that the map Φ is well-defined.

Lemma 5.3. *Under the Hochschild-Kostant-Rosenberg isomorphism (2.6), the composition $\zeta \circ \Phi : HH_\bullet(A_{l,n}) \rightarrow HH_\bullet(A_{l,n})$ is identified with the map C^{-1} of (5.3).*

Proof. Both sides are compatible with the Künneth isomorphism, so that it suffices to consider the case $n = 1$, $l = 0, 1$. Moreover, the natural maps $HH_\bullet(A_{1,1}) \rightarrow HH_\bullet(A_{0,1})$ are injective, and maps C^{-1} and Φ for the algebra $A_{1,1}$ are obtained by restriction from the corresponding maps for the algebra $A_1 = A_{0,1}$. Thus it suffices to consider the algebra $A = A_1 = k[t, t^{-1}] = k[\mathbb{Z}]$. The Hochschild homology $HH_\bullet(A)$ is then only non-trivial in degrees 0 and 1. The i -th term $CH_i(A)$ of the Hochschild complex of the algebra A is given by

$$CH_i(A) = A^{\otimes i+1} = k[\mathbb{Z}^{i+1}],$$

and using the explicit form (2.5) of the isomorphism h_l of Lemma 1.2, one immediately checks that on $CH_i(A)$, the composition $\zeta \circ \Phi$ is given by

$$\zeta(\Phi(\langle g_0, \dots, g_i \rangle)) = \langle g_0 + (p-1)(g_0 + \dots + g_i), g_1, \dots, g_i \rangle$$

for any basis element $\langle g_0, \dots, g_i \rangle \in \mathbb{Z}^{i+1}$. For $i = 0$ and $i = 1$, this reads as

$$\begin{aligned} \zeta(\Phi(t^n)) &= t^{pn}, \\ \zeta(\Phi(\langle t^{n_0}, t^{n_1} \rangle)) &= \langle t^{pn_0+(p-1)n_1}, t^{n_1} \rangle. \end{aligned}$$

This obviously coincides with C^{-1} in degree 0, and since the Hochschild-Kostant-Rosenberg isomorphism sends $\langle t^{n_0}, t^{n_1} \rangle$ to $n_1 t^{n_1+n_0-1} dt$, we also get the result in degree 1. \square

Lemma 5.4. *For any $n \geq 1$, $0 \leq l \leq n$, Proposition 5.1 holds for the algebra $A = A_{l,n}$ of (5.4).*

Proof. The classical Cartier isomorphism and the inverse map C^{-1} give a splitting

$$zHH_\bullet(A) = bHH_\bullet(A) \oplus C^{-1}(HH_\bullet(A)),$$

and by Lemma 5.3, C^{-1} factors through ζ , so that by (4.8), we have $zHH_{\bullet}(A) = z'HH_{\bullet}(A)$. Moreover, since C^{-1} is injective, ζ is injective on the direct summand $\Phi(HH_{\bullet}(A))$ of the decomposition (5.2). Assume that the natural map $\beta : HH_i(A) \rightarrow BHH_{i+1}(A)$ of (4.6) is surjective for $i < l$ for some integer $l \geq 0$. Then we have

$$(5.5) \quad BHH_i(A) \cong HH_{i-1}(A)/zHH_{i-1}(A) \cong bHH_i(A)$$

for $i \leq l$, so that ζ is injective on $\xi(BHH_i(A))$ and $bHH_i(A) = b'HH_i(A)$ in this range of degrees. Thus for $i \leq l$, ζ is injective on the whole $ZHH_i(A)$, and the connecting homomorphism in (4.7) vanishes, so that β is surjective on $HH_l(A)$, too.

By induction, ξ is injective and β is surjective in all degrees, and we have $BHH_{\bullet}(A) \cong bHH_{\bullet}(A)$, $ZHH_{\bullet}(A) \cong zHH_{\bullet}(A)$. It remains to check that the classical Cartier map and the non-commutative Cartier map give the same identification $zHH_{\bullet}(A)/bHH_{\bullet}(A) \cong HH_{\bullet}(A)$. This immediately follows from Lemma 5.3, since they have the same inverse. \square

Proof of Proposition 5.1. By assumption, A is finitely generated, $X = \text{Spec } A$ is smooth, and $HH_{\bullet}(A) = \Omega^*(X)$ is spanned by differential forms $f_0 df_1 \wedge \cdots \wedge df_i$, $f_0, \dots, f_i \in A$. Every such form is the image of the standard form

$$\tau_i = t_0 dt_1 \wedge \cdots \wedge dt_i$$

under the map $\alpha : A_{i,i} = k[t_0, \dots, t_i] \rightarrow A$ sending t_j to f_j , $0 \leq j \leq i$. By Lemma 5.4, this immediately implies that the non-commutative Cartier map $C : ZHH_{\bullet}(A) \rightarrow HH_{\bullet}(A)$ is surjective, so that $\xi : BHH_{\bullet}(A) \rightarrow ZHH_{\bullet}(A)$ is injective. Moreover, fix an element $\tilde{\tau}_i \in ZHH_i(A_{i,i}) = zHH_i(A_{i,i}) \subset HH_i(A_{i,i})$ such that $C(\tilde{\tau}_i) = \tau_i$. Then by the classic Cartier isomorphism, every closed form $a \in Z^i(X) \cong zHH_i(A)$ can be expressed as a finite sum

$$a = \bar{a} + \sum_j \alpha_j(\tilde{\tau}_i),$$

where $\bar{a} \in B^i(X) \cong bHH_i(A)$ is an exact form, and $\alpha_j : A_{i,i} \rightarrow A$ are some algebra maps. Since $\tilde{\tau}_i$ lies in $z'HH_i(A_{i,i}) = zHH_i(A_{i,i})$, a must also lie in $z'HH_i(A)$ by (4.8), so that in fact $z'HH_{\bullet}(A) = zHH_{\bullet}(A)$.

Now, as in the proof of Lemma 5.4, assume by induction that the natural map $\beta : HH_i(A) \rightarrow BHH_{i+1}(A)$ of (4.6) is surjective for $i < l$ for some integer $l \geq 0$. Then again, we have (5.5) for $i \leq l$, so that ζ is injective on $\xi(BHH_i(A))$ and $bHH_i(A) = b'HH_i(A)$ in this range of degrees. Then to finish the proof, it suffices to show that ζ is injective on the whole $ZHH_i(A)$.

Indeed, assume given some class $a \in ZHH_i(A)$ such that $\zeta(a) = 0$. Then we can choose a finite sum decomposition

$$C(a) = \sum_j \alpha_j(\tau_i)$$

for some maps $\alpha_j : A_{i,i} \rightarrow A$. Let

$$a' = a - \sum_j \alpha_j(\tilde{\tau}_i),$$

where $\tilde{\tau}_i \in ZHH_i(A_{i,i}) \subset HH_i(A_{i,i})$ is our fixed lifting of τ_i . Then $C(a') = 0$, so that a' lies in $\xi(BHH_i(A)) \subset ZHH_i(A)$, and therefore $\zeta(a')$ lies in $b'HH_i(A) = bHH_i(A)$. But since $\zeta(a) = 0$, this implies that

$$\bar{a} = \sum_j \zeta(\alpha_j(\tilde{\tau}_i)) = \zeta(a - a')$$

lies in $bHH_*(A) = b'HH_i(A) \subset HH_i(A)$, so that if we denote by C_{cl} the classical Cartier map for the algebra A , we have

$$0 = C_{cl}(\bar{a}) = \sum_j C_{cl}(\alpha_j(\zeta(\tilde{\tau}_i))) = \sum_j \alpha_j(C(\tilde{\tau}_i)) = \sum_j \alpha_j(\tau_i) = C(a).$$

Thus a lies in $\xi(BHH_i(A)) \subset ZHH_i(A)$. Since we already know that ζ is injective on $\xi(BHH_i(A))$, we have $a = 0$. \square

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